

Complex Valued Harmonic Univalent and Multivalent Functions and its Applications

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ABSTRACT : In this paper we will study about how the harmonic function have taken birth, starting from basic definition and theorem of harmonic starlike and convex function, harmonic univalent and multivalent function which is very useful for new researchers .

With the help of complex valued harmonic univalent and multivalent function we will obtain the coefficient inequality, distortion bounds, extreme points and convolution property etc.

The theory of harmonic univalent and multivalent function pull the attention of complex analyst after the basic paper has appeared by Clunie and Sheil Small in 1984. Then after that this theory was developing very fastly and it became important branch of complex analysis.

I. INTRODUCTION

Birth of Harmonic function

The theory of Harmonic univalent and multivalent functions attracted attention of complex analysts after the appearance of a basic paper by Clunie and Sheil Small [15] in 1984. Since then the theory has rapidly developed and became an active branch of complex analysis for researchers

G. Bernard Riemann showed that there always exists an analytic function f that maps a simply connected domain D_1 , onto a simply connected domain D_2 . This original version of the Riemann mapping theorem gave rise to the birth of geometric function theory. Koebe [25] in 1907, discovered a powerful version of the Riemann mapping theorem which state that the functions which are both analytic and univalent in the simply connected domain $D = D_1 \neq C$ maps D onto the open unit disc $\Delta = \{z : |z| < 1\}$ in such a way that $z_0 \in D$, $f(z_0) = z_0$ and $f'(z_0) > 0$. This allows pure and applied mathematicians and engineers to reduce problems about simply connected domains to the special case of the open unit disc Δ .

Harmonic Univalent and Multivalent Function

A univalent (m-valent) function $f(z)$ is characterized by the fact that it is analytic in the unit disc Δ and the equation $f(z) = \omega$ has never more than one (m) solution (s). A class S of univalent functions in which $f(z)$ is normalized with $f(0)=0$ and $f'(0) = 1$ is already widely studied.

Failure to settle the Bieberbach conjecture until 1984 led to the introduction and investigation of several subclasses of S . An important subclass of S denoted by S^* , consists of functions that maps Δ onto a starshaped domain with respect to the origin and another important subclass of S is the family K which maps Δ onto a convex domain.

It is well known that $K \subset S^* \subset S$.

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II. COMPLEX VALUED HARMONIC UNIVALENT AND MULTIVALENT FUNCTION HARMOIC UNIVALENT FUNCTION

The complex valued function $f=u+iv$ and it is continuous and also it is defined in a simply connected complex domain D which is also harmonic in D if both of the u and v are real harmonic in D . Let F and G be analytic in D so that $F(0)=G(0)=0$,

$ReF=Ref=u$, $ReG=Imf=v$ by writing $(F+iG)/2=h$, $(F-iG)/2=g$, The function f can also be represented as $f = h + \bar{g}$, where h and g both are analytic in D . h is known as analytic part of f and g is known as co-analytic part of f . It is observed by Clunie and Sheil Small that the function $f = h + \bar{g}$ is locally univalent and sense-preserving if and only if $|g'(z)| < |h'(z)|$, $z \in D$. Again if f can be normalize so that $f(0)=h(0)=f_z(0)-1=0$. The family of all harmonic, complex valued, orientation-preserving normalized univalent functions defined on Δ is denoted by SH and the function f in SH can be represented as $f = h + \bar{g}$, where,

$$(1) \quad h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \text{ and } g(z) = \sum_{n=1}^{\infty} b_n z^n; \quad |b_1| < 1$$

are analytic functions in Δ .

It follows from the orientation-preserving property that $|b_1| < 1$. Therefore, $(f - \overline{b_1 f}) / (1 - |b_1|^2) \in SH$ whenever $f \in SH$. Thus a subclass SH^0 of SH is defined by $SH^0 = \{f \in SH : g'(0) = b_1 = 0\}$.

Note that $S \subset SH^0 \subset SH$. SH and SH^0 are the families and both are normal families. That is every sequence of functions in SH (or SH^0) has a subsequence that converges locally uniformly in Δ .

It is noted that $SH \equiv S$ if $g=0$.

Let TH denote the sub class of SH with negative coefficients whose members $f = h + \bar{g}$ where h and g are of the form

$$(2) \quad h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \text{ and } g(z) = \sum_{n=1}^{\infty} |b_n| z^n, \quad |b_1| < 1, \quad z \in \Delta$$

COMPLEX VALUED HARMONIC MULTIVALENT FUNCTION:

Suppose f is a harmonic function which is define in a Jordan domain D with boundary C and let f is continuous in \bar{D} and $f(z) \neq 0$ on C . Also let f has no singular zeros in D , and let m to be sum of the orders of the zeros of f in D . Then $\Delta_c \arg(f(z)) = 2\pi m$, where $\Delta_c \arg(f(z))$ denotes the change in argument of $f(z)$ as z traverses C .

It is also proved that if f is sense-preserving harmonic function near a point Z_0 , where $f(Z_0) = \omega_0$ and if $f(z) - \omega_0$ has a zero of order m ($m \geq 1$) at Z_0 , then to each sufficiently small $\epsilon > 0$ there corresponds a $\delta > 0$ with the property: "for each $\alpha \in N_\delta(\omega_0) = \{\omega : |\omega - \omega_0| < \delta\}$, the function $f(z) - \alpha$ has exactly m zeros, counted according to multiplicity, in $N_\epsilon(Z_0)$ ". In particular, f has the open mapping property that is, it carries open sets to open sets.

Let Δ be the open unit disc $\Delta = \{z : |z| < 1\}$ also let $a_k = b_k = 0$ for $0 \leq k < m$ and $a_m = 1$. Certain subclasses of the family $SH(m)$ was introduced and studied by Ahuja and Jahangiri, $m \geq 1$ of all multivalent harmonic and orientation preserving functions in Δ . A function f in $SH(m)$ can also be represented as $f = h + \bar{g}$, where h and g are

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$$h(z) = z^m + \sum_{n=2}^{\infty} a_{n+m-1} z^{n+m-1} \qquad g(z) = \sum_{n=1}^{\infty} b_{n+m-1} z^{n+m-1}, \quad |b_m| < 1.$$

Functions in SH(m) are also harmonic and sense-preserving in Δ if $J_f > 0$ in Δ . Clunie and Sheil Small study in detail about the class SH(1) of harmonic univalent functions. It was observed that multivalent mapping does not need be orientation-preserving.

Let TH(m) show the subclass of SH(m) whose members are of the form

$$h(z) = z^m - \sum_{n=2}^{\infty} |a_{n+m-1}| z^{n+m-1} \qquad g(z) = \sum_{n=1}^{\infty} |b_{n+m-1}| z^{n+m-1}, \quad |b_m| < 1.$$

III. PROPERTIES OF COMPLEX VALUED HARMONIC UNIVALENT AND MULTIVALENT FUNCTIONS

In this Paper we discuss some properties of complex valued harmonic univalent and multivalent function like coefficient inequalities, distortion bounds, extreme points, neighbourhood properties, radius of convexity, radius of starlikeness etc.

A harmonic mapping $f \in SH$ (or $f \in SH^0$) which is also sense preserving is known as harmonic starlike mapping in Δ if the range $f(\Delta)$ is starlike with respect to the origin. Similarly a function $f \in SH$ (or $f \in SH^0$) is known as harmonic convex in Δ if $f(\Delta)$ is a convex domain.

COEFFICIENT INEQUALITIES

Theorem 1

For a function $f(z) \in SH$ to be harmonic starlike univalent in Δ , the necessary and sufficient condition is that

$$(1) \quad \frac{\partial}{\partial \theta} \{ \arg f(re^{i\theta}) \} > 0, \quad z \in \Delta, \quad (z = re^{i\theta}, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq 1).$$

S^*H Show the class of harmonic starlike univalent functions.

Note that if $b_1 = 0$ then $S^*H \equiv S^*H^0$.

T^*H Show the class of harmonic starlike univalent functions $f \in TH$.

Theorem 2

For a function $f(z) \in SH$ to be harmonic convex univalent in Δ , the necessary and sufficient condition is that

$$(2) \quad \frac{\partial}{\partial \theta} \left\{ \arg \left(\frac{\partial}{\partial \theta} \arg f(re^{i\theta}) \right) \right\} > 0, \quad z \in \Delta, \quad (z = re^{i\theta}, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq 1).$$

The class of harmonic convex univalent functions is shown as KH.

Note that if $b_1 = 0$ then $KH \equiv KH^0$.

T^*KH Show the class of harmonic convex univalent functions.

Theorem 3

If $f = h + \bar{g}$ then it satisfies

$$(3) \quad \sum_{n=2}^{\infty} n |a_n| + \sum_{n=1}^{\infty} n |b_n| \leq 1$$

Where h and g are defined

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then, $f \in S^*H$.

Theorem 4

If $f = h + \bar{g}$, then it satisfies

$$(4) \quad \sum_{n=2}^{\infty} n^2 |a_n| + \sum_{n=1}^{\infty} n^2 |b_n| \leq 1$$

Where h and g are defined

then, $f \in KH$.

Theorem 5

Let $f = h + \bar{g}$ then $f \in T^*H$ if and only if

$$(5) \quad \sum_{n=2}^{\infty} n |a_n| + \sum_{n=1}^{\infty} n |b_n| \leq 1.$$

Theorem 6

Let $f = h + \bar{g}$, then $f \in KH$ if and only if

$$(6) \quad \sum_{n=2}^{\infty} n^2 |a_n| + \sum_{n=1}^{\infty} n^2 |b_n| \leq 1.$$

By these conditions, various extremal properties, extreme points, distortion bounds, covering theorems, convolution properties for the classes T^*H and KH have been defined.

Extreme points A function $f(z)$ is known as an extreme point of G in a family G if $f(z)$ cannot be represented as a proper convex combination of two different functions in G .

Radius of Starlikeness

The radius of largest circle $|z|=r$ such that the image of $|z|=r$ for all harmonic functions $f(z)$ in a given class always encloses a starlike domain is called the radius of starlikeness of that classes.

Radius of convexity

The radius of largest circle $|z|=r$ such that the image of $|z|=r$ for all harmonic functions $f(z)$ in a given class always encloses a convex domain is called the radius of convexity of that class.

MULTIPLIER FAMILY

The multipliers $\{c_{n+m-1}\}$ and $\{d_{n+m-1}\}$ provide a transition from multivalent harmonic starlike functions to multivalent harmonic convex functions, include many more subclasses of $SH(m)$ and $SH(1)$. For example.

$$F_m \left(\left\{ \frac{n-1+m(1-\alpha)}{1-\alpha} \right\}, \left\{ \frac{n-1+m(1+\alpha)}{1-\alpha} \right\} \right) \equiv TH(m, \alpha)$$

$$F_m \left(\left\{ \frac{(n+m-1)(n-1+m(1-\alpha))}{m(1-\alpha)} \right\}, \left\{ \frac{(n+m-1)(n-1+m(1+\alpha))}{m(1-\alpha)} \right\} \right) \equiv KH_m(\alpha)$$

$$F_m(\{n+m-1\}, \{n+m-1\}) \equiv TH(m, 0) := TH(m)$$

$$F_m \left(\left\{ \frac{(n+m-1)^2}{m} \right\}, \left\{ \frac{(n+m-1)^2}{m} \right\} \right) \equiv KH_m(0) := KH_m$$

$$F_1(\{n\}, \{n\}) \equiv TH(1, 0) \equiv TH$$

$$F_1(\{n^2\}, \{n^2\}) \equiv KH(1, 0) \equiv KH$$

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$$F_1(\{n^p\}, \{n^p\}) := F(\{n^p\}, \{n^p\}) \quad p > 0$$

$$F_1(\{c_n\}, \{d_n\}) = F(\{c_n\}, \{d_n\}).$$

Guney and Ahuja [19] obtained the following results for the multiplier family $F_m(\{c_{n+m-1}\}, \{d_{n+m-1}\})$:

NEIGHBOURHOODS

Rushcheweyh [31] gave the following definition of neighbourhood.

The δ – neighbourhood of $f = h + \bar{g}$ with h and g are of the form (1) is defined as the set

$$N_\delta(f) = \left\{ F : F(z) = z - \sum_{n=2}^{\infty} |A_n| z^n + \sum_{n=1}^{\infty} |B_n| \bar{z}^n \text{ and } \sum_{n=1}^{\infty} n(|a_n - A_n| + |b_n - B_n|) \leq \delta, \delta > 0 \right\}.$$

IV. APPLICATIONS OF COMPLEX VALUED HARMONIC UNIVALENT AND MULTIVALENT FUNCTION

There are following applications of harmonic functions:

1) SOLUTION OF LAPLACE'S EQUATION

Harmonic functions are used as a solution of Laplace's equation arise from the resulting prevention of spurious local minima.

2) APPLICATIONS TO ROBOTICS

Harmonic function are used in robotics application such as fast surface normal computation, completeness ability to exhibit different modes of behaviour and robust control in the presence of unanticipated obstacle and errors. A system that demonstrate this usefulness for control of revolute robot arm is discussed.

3) ADVANTAGE FOR POTENTIAL FIELD PATH PLANNING

Harmonic function are used as advantage for potential field path planning because they do not exhibit spurious local minima. Harmonic function also give a complete path planning algorithm.

4) BASIS FOR A REACTIVE ADMITTANCE CONTROL

Harmonic function can be used as the basis for a reactive admittance control. Such property allow incremental updating of the environmental model. Method for computing harmonic function reply well to the sensed changes in the environment and can be used for control while the environmental model is being updated.

5) APPLICATION TO SCALAR AND VECTOR QUANTITIES

Harmonic functions have been applied to scalar quantities such as temperature and conserved potentials in physical sciences, also it is valid to basic properties to vectors such as electrostatic field, magnetostatic field, current density, and gravitational force. We find these physical fields as Laplace fields and develop the Laplace field theory. Based on this, a spherical mean value method is proposed to decrease the noise in the measurement of these fields. Methods to quantify related variables such as magnetic susceptibility under in homogeneous field conditions for objects of arbitrary shapes are also developed. Nuclear magnetic resonance phase imaging (MRPI) can non-invasively map the three-dimensional magnetic field z component and the temperature distribution. We show theoretically and experimentally that the MR phase map is a Laplace field and employ it as a convenient basis for the validation of the Laplace field theory and methodology. We implement the spherical mean value method to map out the magnetic field (z component) and relative temperature change with high precision, i.e., $10^{-11} \sim 10^{-12}$ Tesla for magnetic field and $10^{-3} \sim 10^{-4}$ °C for temperature. The random noise is reduced by over three orders of magnitude without cost of any extra experimental time. Furthermore, with MRPI, we measure the magnetic susceptibilities for $CuSO_4$ -solution phantoms of irregular shapes without the presumption of the homogeneity of the applied field. Numerical simulations of the boundary value problem of Laplace equation extend the methodology of precision improvement and magnetic susceptibility quantization to nonspherical boundaries. The methodology developed in this work can be generalized to applications for any physical fields whose distributions in space are Laplace fields.

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